Communication Avoiding and Overlapping for Numerical Linear Algebra

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   - Conclusions

Communication is Expensive

Communication has two components:

- **Bandwidth cost:** # of words moved / bandwidth
- **Latency cost:** # messages $\times$ latency

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Communication exists in **memory hierarchy** and **network**
Things are bad and **getting worse**:

$$\textit{flop time} \ll 1/\textit{bandwidth} \ll \textit{latency}$$

Annual improvements [FOSC]			
Flop time		Bandwidth	Latency
59%	Network	26%	15%
	DRAM	23%	5%

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Communication is also expensive in **energy**

Communication Avoidance vs Overlapping

Two techniques to minimize the impact of communication:

Communication Avoidance vs Overlapping

Two techniques to minimize the impact of communication:

1. **Communication Avoidance (CA):**

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2. **Communication Overlapping (CO):**
   - Reduces the impact of each communication event by overlapping it with computation
   - Hides bandwidth and/or latency cost
   - Most effective if communication & computation are balanced

CA and CO in Linear Algebra

- We studied these techniques in three Linear Algebra routines:
  - Matrix Multiplication (SUMMA and Cannon's algorithms)
  - Cholesky factorization
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Optimizations / Algorithm	Overlapping	Avoidance	Overlapping & Avoidance
SUMMA	PRIOR	PRIOR	
Cannon's	PRIOR	PRIOR	
Cholesky	PRIOR		
TRSM	PRIOR		

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Algorithm \ Optimizations	Overlapping	Avoidance	Overlapping & Avoidance
SUMMA	PRIOR	PRIOR	NEW
Cannon's	PRIOR NEW: One sided communication	PRIOR NEW: One sided communication	NEW
Cholesky	PRIOR	NEW	NEW
TRSM	PRIOR	NEW*	NEW*

*Uses replication but not communication optimal

CA and CO in Linear Algebra

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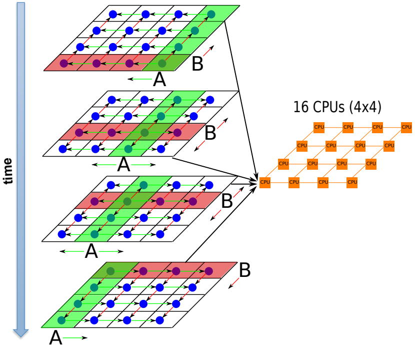
2D Matrix Multiplication (SUMMA)

[Van De Geijn and Watts 97]



- Outer product form of Mat Mul
- Partitions A, B and C in 2 dimensions
- Row and column broadcast on 2D grid
- Costs:
  - $O(n^3/p)$ flops
  - $O(n^2/\sqrt{p})$ words moved
  - $O(\sqrt{p})$ messages

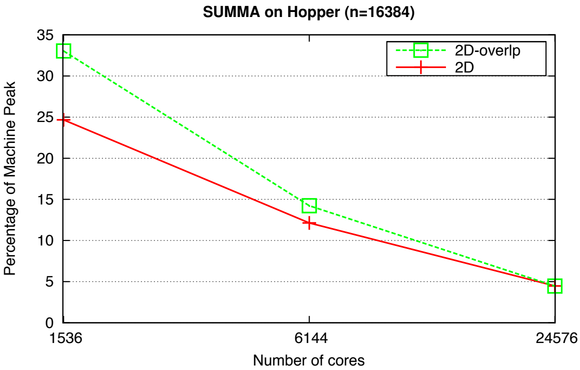
2D Matrix Multiplication with CO (SUMMA)

- We overlap the broadcasts of next iteration with the local Mat.Mul computation of current iteration
- Theoretically the execution time becomes $t_{exec} = O(\max(t_{computation}, t_{communication}))$
- If communication and computation are balanced we can achieve up to $2\times$ speedup
- **Additional communication buffers are needed**

Experimental setup

- Experiments on Hopper, a Cray XE6 system (153,216 cores)
- We will focus on communication-limited problems (i.e. small problems on large machine configurations)
- Strong scaling experiments

Overlapping yields more benefits at medium scale



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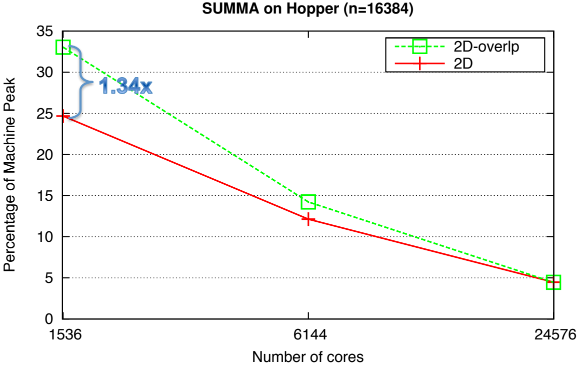
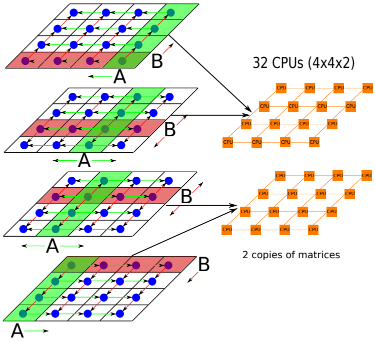


- At medium scale where communication and computation are balanced we observe larger benefits (1.34$\times$ speedup)
- At large scale overlapping does not help

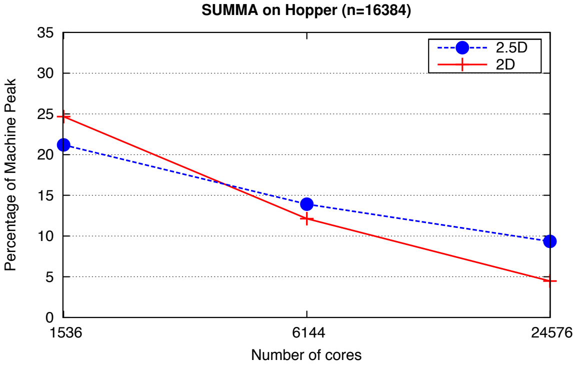
2.5D Matrix Multiplication (SUMMA)

[McColl and Tiskin 99], [Solomonik and Demmel 11]



- The 2.5D algorithm uses extra memory to reduce communication
- Each one of the c layers of processors computes a different contribution to the matrix C
- Works for c copies, $c \in [1, p^{1/3}]$
- Costs:
  - $O(n^3/p)$ flops
  - $O(n^2/\sqrt{c \cdot p})$ words moved
  - $O(\sqrt{p/c^3})$ messages

Communication avoidance helps a lot at large scale



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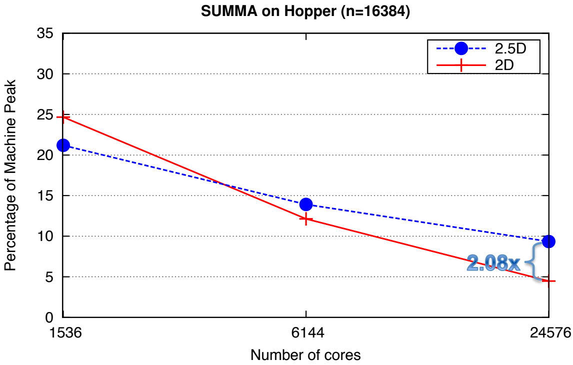
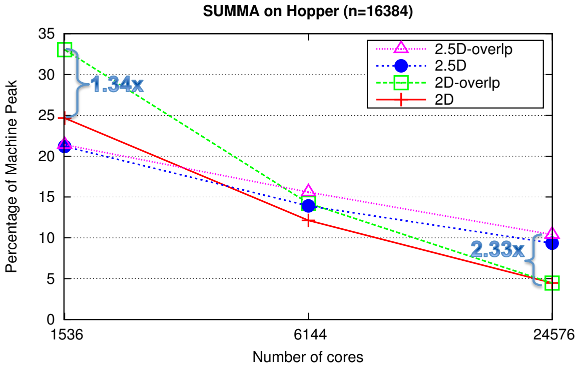


- At large scale avoidance helps a lot (2.08$\times$ speedup) (there is a lot of communication to avoid!)
- At medium scale communication avoidance may yield slowdown

2.5D Matrix Multiplication with CO (SUMMA)

- We overlap the broadcasts of the next iteration with the local Mat.Mul computation of current iteration **on each** of the c processor layers
- **Additional communication buffers are needed on top of the extra memory needed for replication**

Putting everything together



- **At medium scale overlapping itself yields best performance**
- **At large scale combining both techniques pays off**

Cannon's algorithm

- In SUMMA we use collective communication operations

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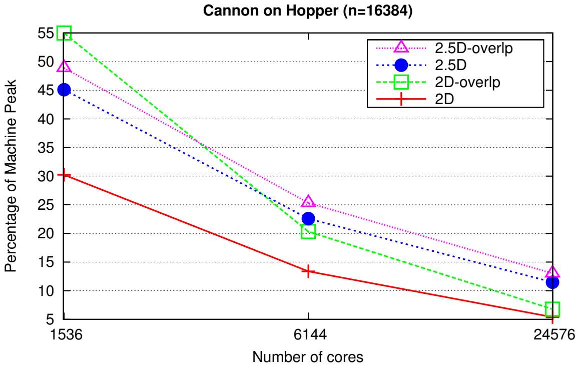
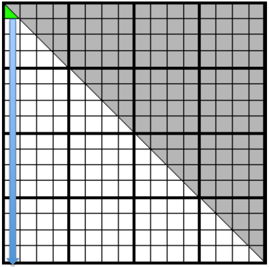


2D Cholesky factorization

- Factorize a symmetric positive definite matrix A into $A = L \cdot L^T$, where L is lower triangular
- Take advantage of symmetry and store only half of matrix A
- Employ block-cyclic layout for load-balance

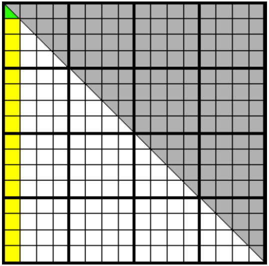
2D Cholesky factorization



1. Factorize the upper-left (green) block & broadcast it to the column
2. Update via TRSM the (yellow) block column
3. Broadcast the factorized column in two phases & update the trailing matrix (white blocks)
4. Continue with the factorization of the second block column and repeat previous steps until all matrix is factorized

- Communication involved is row and column broadcasts
- There are dependencies among rows and column during factorization

2D Cholesky factorization



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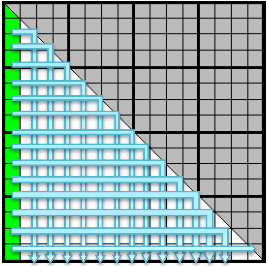

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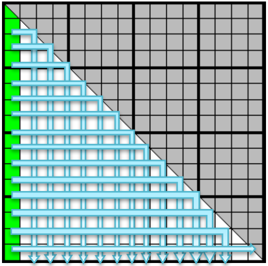

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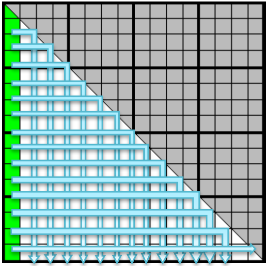

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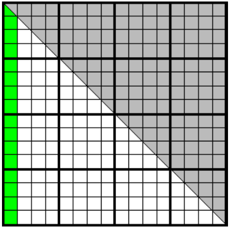
2D Cholesky factorization with overlapping



1. Factorize the 1^{st} block-column
2. Broadcast factorized column
3. Update & factorize **only** the 2^{nd} block-column
4. **Overlap**
   - broadcast of the 2^{nd} block-column
   - update of the rest trailing matrix (using 1^{st} block-column)

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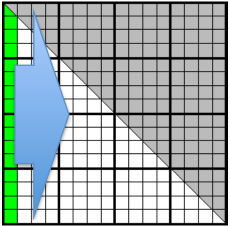

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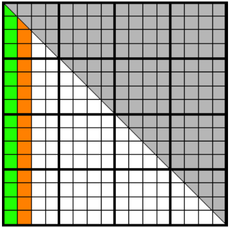

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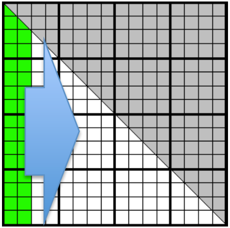

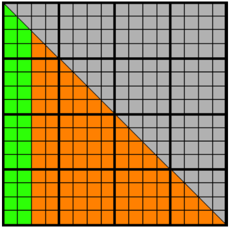

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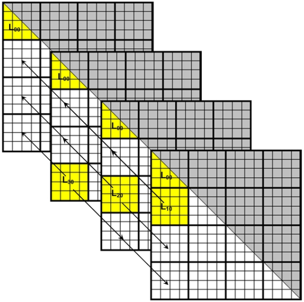
2.5D Cholesky factorization

[McColl and Tiskin 99], [Solomonik and Demmel 11]

- Employ two levels of blocking: “Fat panels” and “blocks”



1. All layers jointly factorize a “fat” panel
2. Broadcast different subpanels within each layer & update trailing matrices
3. All-reduce the next ”fat” panel to accumulate the updates

2.5D Cholesky factorization

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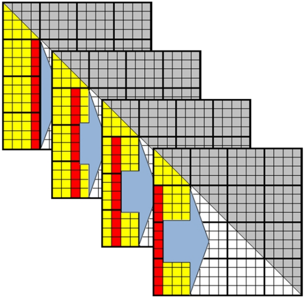

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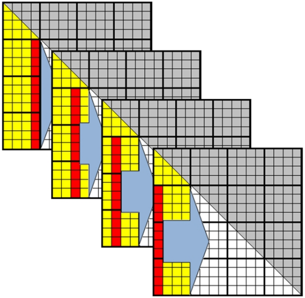

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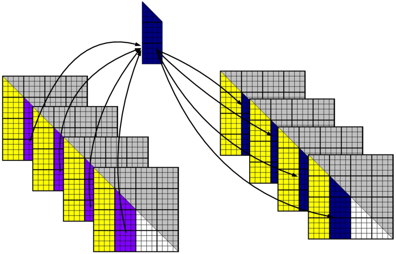

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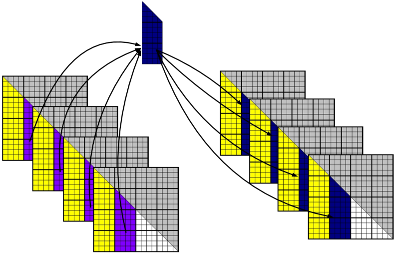

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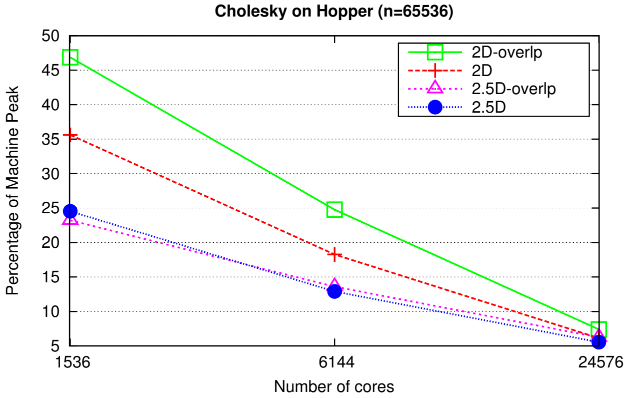
- At step 2 we can overlap computation and communication similarly to the 2D version

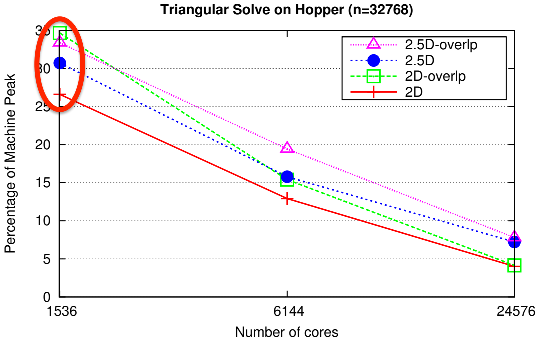
Performance results on Hopper (Cray XE6)



- CO helps more at the smallest scale (1,536 cores)
- Have not reached yet the cross-point of CA and 2D
- Future work: Aggregate updates to improve performance

Triangular Solve (TRSM)

- Computes X, such that $X \cdot U = B$ with upper-triangular U
- Similar parallelization to Cholesky



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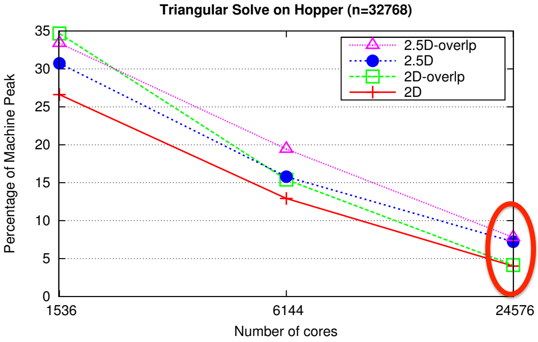

- At small scale overlapping outperforms other versions
- At large scale the combining optimizations is the best option

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replication ↑	↑/↓	↑/↓	NA
fat panel size ↑	↓	↓	NA

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- Communication performance depends on number of processors
- **Given a problem instance and a machine configuration would like to predict the optimal variant**

Methodology for constructing performance models

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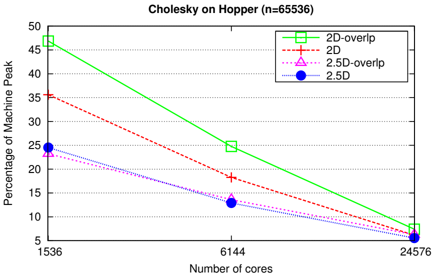
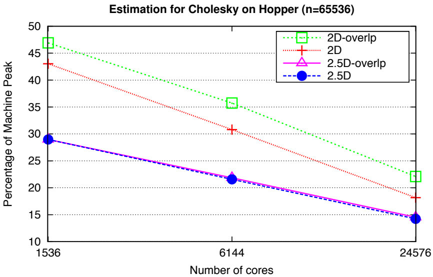
  - Estimate computation times through BLAS microbenchmarks
  - Estimate communication times through LogGP model
- We take into account possible idle times

Models make correct qualitative predictions





- **The models predict correctly the performance ranking of the four variants**
- They encapsulate the complicated interactions
- They can take into account memory limitations
- Ongoing work shows that we can predict absolute performance accurately through integration of congestion

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   - We developed detailed performance models
   - They encapsulate complicated interactions between parameters
   - They predict correctly the performance ranking

Thank you!